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Van Kampen theorems for categories of covering morphisms in lextensive categories[1]

Ronald Brown[a,*], George Janelidze[b]

[a] School of Mathematics, University of Wales, Bangor, Gwynedd LL57 1UT, UK

[b] Mathematics Institute, Georgian Academy of Sciences, Tbilisi, Georgia

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Abstract

We give a form of the Van Kampen Theorem involving covering morphisms in a lextensive category. This includes the usual results for covering maps of locally connected spaces, for light maps of compact Hausdorff spaces, and for locally strong separable algebras. © 1997 Elsevier Science B.V.

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0. Introduction

Let B be a topological space and B_1, B_2 open subspaces of B such that $B = B_1 \cup B_2$. The classical Van Kampen Theorem asserts that the canonical morphism of groups

$$\pi_1(B_1, b) +_{\pi_1(B_1 \cap B_2, b)} \pi_1(B_2, b) \to \pi_1(B, b) \tag{1}$$

is an isomorphism, i.e., that the fundamental group $\pi_1(B,b)$ is a pushout of $\pi_1(B_1,b)$ and $\pi_1(B_2,b)$ over $\pi_1(B_1 \cap B_2, b)$. Here it is assumed that there is a single base point b, say, contained in $B_1 \cap B_2$, which is itself assumed path-connected.

A common proof of this theorem uses the definition of the fundamental group in terms of paths. However, it is well known that there is another proof of the isomorphism (1), for the case of $B_1 \cap B_2$ connected, using covering spaces. This is sometimes called the "tautologous proof"; an exposition is given in [7, Section 4.6].

However, for the isomorphism (1) one can avoid all these assumptions of connectivity and the choice of base point by replacing the fundamental groups $\pi_1(-,b)$ by the

* Corresponding author. E-mail: r.brown@bangor.ac.uk.

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fundamental groupoids $\pi_1(-)$ – see [2, 3]. The proof of this theorem in [2, 3] uses the definition of the fundamental groupoid in terms of paths. There is an analogous result in terms of covering spaces, as follows.

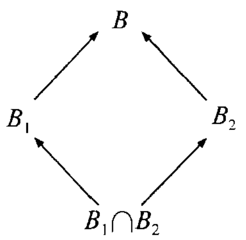
Suppose that B is a "good" space – so that the fundamental groupoid $\pi_1(B)$ classifies the coverings of B, i.e., there is an equivalence of categories (see e.g. [3, Ch. 9])

$$\text{Coverings}(B) \sim \text{Sets}^{\pi_1(B)}. \tag{2}$$

Then (1) can be generalised and formulated in terms of coverings as an equivalence of categories

$$\text{Coverings}(B) \sim \text{Coverings}(B_1) \times_{\text{Coverings}(B_1 \cap B_2)} \text{Coverings}(B_2). \tag{3}$$

We will then say that the class of coverings satisfy the Van Kampen Theorem for the given diagram



(4)

In this form we require only that B_1, B_2 form an open cover of B.

We shall put this form of the Van Kampen Theorem in a general setting. One major reason is to consider the theorem for other classes of maps than topological coverings. The second, is that analogous theorems occur in algebra [12], where the category under consideration is the opposite of a category of commutative rings. This algebraic situation is the one from which arose Grothendieck's general formulation of Galois theory, for which he introduced the notion of descent, and this led to the Galois theory in categories of [8]. Thus, one of our aims is to produce one Van Kampen theorem which includes these cases, and others.

We use two concepts which were introduced a long time ago but until recently have not been much studied for their own sakes.

One concept is the condition on a map of spaces that it be an effective descent morphism. This condition on the map $B_1 + B_2 \to B$ is found to be necessary and sufficient for the Van Kampen Theorem. This contrasts with the sufficient homotopical conditions used for example in [3, Ch. 8].

A second concept is that of lextensive category, which turns out to be exactly the natural context for these questions. The advantage of such categories is that we can prove a theorem which includes both the topological and algebraic situations which are of immediate interest, as well as many others. These examples are discussed in the final section.

1. A general setting for the Van Kampen theorems

Here is the general setting for the above considerations.

Let $\mathbb{C}$ be a category with pullbacks, and let $\mathscr{F}$ be a class of morphisms of $\mathbb{C}$ which contains all isomorphisms and is pullback stable. The latter means that if

$$\begin{array}{ccc} E \times_B A & \xrightarrow{pr_2} & A \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B \end{array} \tag{5}$$

is a pullback diagram in $\mathbb{C}$ with α in $\mathscr{F}$, then pr_1 is also in $\mathscr{F}$.

Now let $\mathscr{F}(B)$ be the full subcategory of the comma category $(\mathbb{C} \downarrow B)$ with objects all pairs (A, α) with $\alpha : A \to B$ in $\mathscr{F}$. Any morphism $p : E \to B$ in $\mathbb{C}$ determines the pullback functor

$$p^* : \mathscr{F}(B) \to \mathscr{F}(E) \tag{6}$$

defined by

$$p^*(A, \alpha) = (E \times_B A, pr_1), \tag{7}$$

via the pullback (5).

Now consider a commutative diagram in $\mathbb{C}$ of the form

$$\begin{array}{ccccc} & & B & & \\ & {\scriptstyle g_1}\nearrow & & \nwarrow{\scriptstyle g_2} & \\ B_1 & & & & B_2 \\ & {\scriptstyle f_1}\nwarrow & & \nearrow{\scriptstyle f_2} & \\ & & B_0 & & \end{array} \tag{8}$$

We define $\mathscr{F}(B_1) \times_{\mathscr{F}(B_0)} \mathscr{F}(B_2)$ as the category of triples $((A_1, \alpha_1), (A_2, \alpha_2), \phi)$ where (A_1, α_1) is in $\mathscr{F}(B_1)$, (A_2, α_2) is in $\mathscr{F}(B_2)$, and $\phi : f_1^*(A_1, \alpha_1) \to f_2^*(A_2, \alpha_2)$ is an isomorphism. Since $g_1 f_1 = g_2 f_2$, the pair (g_1, g_2) induces a functor

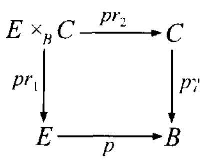
$$K_{g_1, g_2} : \mathscr{F}(B) \to \mathscr{F}(B_1) \times_{\mathscr{F}(B_0)} \mathscr{F}(B_2). \tag{9}$$

Definition 1.1. We will say that $\mathscr{F}$ satisfies the Van Kampen Theorem for a given commutative diagram (8) if the functor K_{g_1, g_2} above is an equivalence of categories.

Accordingly, by "Van Kampen Theorems" we will mean assertions that for certain diagrams (8), K_{g_1, g_2} is an equivalence of categories.

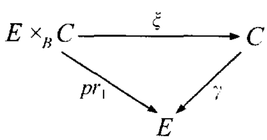
2. Effective descent morphisms

Let $\mathbb{C}$, $\mathscr{F}$ and $p : E \to B$ be as above. An *$\mathscr{F}$-descent data for p* is a triple (C, γ, ξ) in which (C, γ) is an object in $\mathscr{F}(E)$ and $\xi : E \times_B C \to C$ is a morphism in $\mathbb{C}$ from the pullback



(10)

to C making the following diagrams commute:



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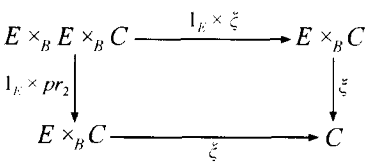

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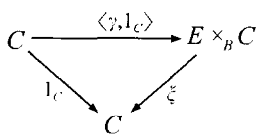

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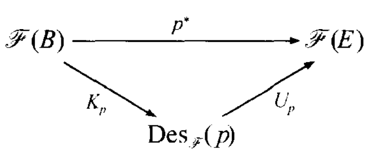
There is an obvious notion of morphism of $\mathscr{F}$-descent datas for p, which form a category $\mathrm{Des}_{\mathscr{F}}(p)$. There is a commutative diagram



(14)

where $U_p : (C, \gamma, \xi) \mapsto (C, \gamma)$ is the forgetful functor, and K_p, the so-called *comparison functor*, is defined by $(A, \alpha) \mapsto (E \times_B A, pr_1, \xi)$, where $E \times_B A$ and pr_1 are as in the pullback (5), and $\xi = 1_E \times pr_2 : E \times_B E \times_B A \to E \times_B A$.

The morphism $p : E \to B$ is said to be an *effective $\mathscr{F}$-descent morphism* if the functor K_p is an equivalence of categories (this and other related notions are recalled in detail in [10]). Following [10], we will say that p is an effective global-descent morphism if this is true for $\mathscr{F}$ being the class of all morphisms in $\mathbb{C}$. Recall that in this case $\mathrm{Des}_{\mathscr{F}}(p) = (\mathbb{C} \downarrow E)^T$ where T is the monad determined by the adjunction

$$(\mathbb{C} \downarrow B) \underset{p_!}{\overset{p^*}{\rightleftarrows}} (\mathbb{C} \downarrow E) \qquad p_! \dashv p^*, \tag{15}$$

where $p_!$ is the composition with p. That is, p is an effective global-descent morphism if and only if $p^* : (\mathbb{C} \downarrow B) \to (\mathbb{C} \downarrow E)$ is monadic.

The following result, also mentioned in [10], is often useful.

Proposition 2.1. *Suppose that for any pullback* (5) *in* $\mathbb{C}$ *with* $pr_1 : E \times_B A \to E$ *in* $\mathscr{F}$, $\alpha : A \to B$ *is also in* $\mathscr{F}$; *then every effective global-descent morphism in* $\mathbb{C}$ *is an effective* $\mathscr{F}$*-descent morphism.*

3. The general Van Kampen Theorem

A category $\mathbb{C}$ is said to be *lextensive* if it has finite limits (in fact we will use only pullbacks), and finite coproducts which are disjoint and universal; these conditions have been considered by many authors for a long time, but we take the name "lextensive" from a recent paper [6] of Carboni, Lack and Walters, where a nice introduction to the subject is given.

One can also define a category $\mathbb{C}$ to be lextensive if it has finite limits and finite coproducts, and for any pair X, Y of objects of $\mathbb{C}$ the functor

$$(\mathbb{C} \downarrow X) \times (\mathbb{C} \downarrow Y) \xrightarrow{+} (\mathbb{C} \downarrow X + Y) \tag{16}$$

is an equivalence – or, equivalently, the functor

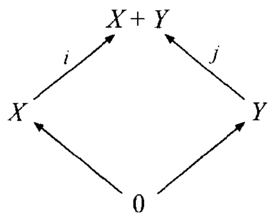
$$(\mathbb{C} \downarrow X + Y) \xrightarrow{\langle i^*, j^* \rangle} (\mathbb{C} \downarrow X) \times (\mathbb{C} \downarrow Y), \tag{17}$$

where $i : X \to X + Y$ and $j : Y \to X + Y$ are the coproduct morphisms, is an equivalence.

Note that if $\mathbb{C}$ is a lextensive category, then $(\mathbb{C} \downarrow 0)$, where 0 is an initial object of $\mathbb{C}$, is a codiscrete groupoid, and so

$$(\mathbb{C} \downarrow X) \times (\mathbb{C} \downarrow Y) \sim (\mathbb{C} \downarrow X) \times_{(\mathbb{C} \downarrow 0)} (\mathbb{C} \downarrow Y).$$

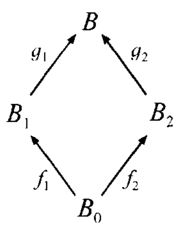
Thus, the equivalence (17) is just a Van Kampen Theorem for the diagram



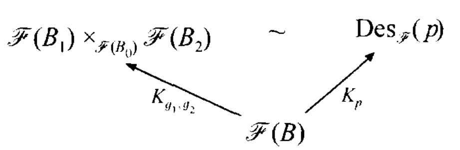
(18)

Moreover, it turns out that any kind of Van Kampen Theorem in a lextensive category is equivalent to a certain "usual" property, namely the property of being an effective descent morphism, as we see from the Proposition 3.2 below.

Lemma 3.1. *Let*



be a pullback diagram in a lextensive category $\mathbb{C}$, *in which* g_1 *and* g_2 (*and so also* f_1 *and* f_2) *are monomorphisms, and let* $\mathscr{F}$ *be a class of morphisms in* $\mathbb{C}$ *which is pullback stable and contains all isomorphisms. Let* $p : B_1 + B_2 \to B$ *denote the morphism induced by* g_1 *and* g_2. *Then there exists an equivalence of categories between* $\mathscr{F}(B_1) \times_{\mathscr{F}(B_0)} \mathscr{F}(B_2)$ *and* $Des_{\mathscr{F}}(p)$ *such that the diagram*



commutes (*up to an isomorphism*).

Proof. We will just show how to construct the desired functor from $\mathrm{Des}_{\mathscr{F}}(p)$ to $\mathscr{F}(B_1) \times_{\mathscr{F}(B_0)} \mathscr{F}(B_2)$ (on objects) and omit the calculations.

Given an object (C, γ, ξ) in $\mathrm{Des}_{\mathscr{F}}(p)$, we first use the equivalences (16) and (17) and obtain

$$B_1 \times_B C \xrightarrow{\xi_1} C_1 \xrightarrow{\gamma_1} B_1,$$

$$B_2 \times_B C \xrightarrow{\xi_2} C_2 \xrightarrow{\gamma_2} B_2,$$

so that $C = C_1+C_2$, $\gamma = \gamma_1+\gamma_2$, and $\xi = \xi_1+\xi_2$ *via* $(B_1+B_2)\times_B C \cong B_1\times_B C+B_2\times_B C$. Then we induce compositions

$$\xi_{ij} : B_i \times_B C_j \to B_i \times_B C \xrightarrow{\xi_i} C_i$$

$i,j = 1,2$, so that ξ is determined by $(\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22})$. Then we note that $B_1 \times_B B_1 \cong B_1$, $B_1 \times_B B_2 \cong B_0 \cong B_2 \times_B B_1$, $B_2 \times_B B_2 \cong B_2$ (the first and the last isomorphisms follow from the fact that g_1 and g_2 are monomorphisms), which gives

$$B_i \times_B C_j \cong \begin{cases} C_i & \text{if } i = j, \\ B_0 \times_{B_j} C_j & \text{if } i \neq j. \end{cases}$$

Moreover, it is easy to show that ξ_{11} and ξ_{22} must be canonical isomorphisms and ξ_{12} and ξ_{21} can be factored as

$$\begin{array}{ccc} B_1 \times_B C_2 & \xrightarrow{\xi_{12}} & C_1 \\ {\scriptstyle\cong}\downarrow & & \downarrow \\ B_0 \times_{B_2} C_2 & \xrightarrow[\xi^*_{12}]{} & B_0 \times_{B_1} C_1, \end{array} \qquad \begin{array}{ccc} B_2 \times_B C_1 & \xrightarrow{\xi_{21}} & C_2 \\ {\scriptstyle\cong}\downarrow & & \downarrow \\ B_0 \times_{B_1} C_1 & \xrightarrow[\xi^*_{21}]{} & B_0 \times_{B_2} C_2; \end{array}$$

in fact one can show also that ξ^*_{12} and ξ^*_{21} are the inverses of each other.

Now the desired functor can be constructed by

$$(C, \gamma, \xi) \mapsto ((C_1, \gamma_1), (C_2, \gamma_2), \xi^*_{21}). \qquad \square$$

According to Descent theory, this lemma gives:

Proposition 3.2 ("General Van Kampen Theorem"). *Let*

$$\begin{array}{ccccc} & & B & & \\ & {\scriptstyle g_1}\nearrow & & \nwarrow{\scriptstyle g_2} & \\ B_1 & & & & B_2 \\ & {\scriptstyle f_1}\nwarrow & & \nearrow{\scriptstyle f_2} & \\ & & B_0 & & \end{array}$$

be a pullback diagram in a lextensive category $\mathbb{C}$ *in which* g_1 *and* g_2 *are monomorphisms. Let* $\mathscr{F}$ *be a class of morphisms in* $\mathbb{C}$ *which is pullback stable and contains all isomorphisms. Then the following are equivalent*:

(a) $\mathscr{F}$ *satisfies the Van Kampen Theorem for the pullback diagram above*;

(b) *the morphism* $B_1 + B_2 \to B$ *induced by* g_1 *and* g_2 *is an effective* $\mathscr{F}$*-descent morphism.*

Note that if $\mathscr{F}$ is the class of all morphisms, then the pullback above is also a pushout. This may be deduced from Proposition 3.2, or shown directly.

4. Examples

Example 4.1. Let $\mathbb{C}$ be a lextensive category and let $\mathscr{F}$ be the class of decidable (= separable) morphisms in $\mathbb{C}$ in the sense of [4]. Then by Proposition 2.1 and [4, Theorem 1.1 (12)] every effective global-descent morphism in $\mathbb{C}$ is an effective $\mathscr{F}$-descent morphism.

Example 4.2. Let

$$\mathbb{C} \underset{H}{\overset{I}{\rightleftarrows}} \mathbb{X}, \quad I \dashv H, \tag{18}$$

be an adjunction between categories $\mathbb{C}$ and $\mathbb{X}$ such that $\mathbb{C}$ is lextensive and for any object C in $\mathbb{C}$, the category $(\mathbb{X} \downarrow I(C))$ is a full reflective subcategory of $(\mathbb{C} \downarrow C)$ *via* the induced adjunction between $(\mathbb{C} \downarrow C)$ and $(\mathbb{X} \downarrow I(C))$. Let $\mathscr{F}$ be the corresponding class of "coverings" in $\mathbb{C}$, i.e., the class of all morphisms $\alpha : A \to B$ in $\mathbb{C}$ such that there exists an effective descent morphism $p : E \to B$ for which the diagram

$$\begin{array}{ccc} E \times_B A & \longrightarrow & HI(E \times_B A) \\ \downarrow & & \downarrow \\ E & \longrightarrow & HI(E) \end{array}$$

is a pullback (see [8] and the references there). Then clearly Proposition 2.1 can be applied and we conclude that every effective global-descent morphism in $\mathbb{C}$ is an effective $\mathscr{F}$-descent morphism.

Example 4.3. Let $\mathbb{C}$ be the category of topological spaces and let $\mathscr{F}$ be the class of all local homeomorphisms; [10, Theorem 4.7] says that every effective global-descent morphism in $\mathbb{C}$ is an effective $\mathscr{F}$-descent morphism.

Observe that the situations of Examples 4.1, 4.2 themselves include several important concrete examples. As mentioned in [4], 4.1 includes the examples of separable algebras and decidable morphisms in a topos. Example 4.2 includes the following: (i) componentially locally strongly separable algebras in the sense of Magid [11], as shown in [8]; (ii) light maps of compact Hausdorff topological spaces (described as categorical coverings in [5]); (iii) coverings of locally connected topological spaces. In this last example, we take $\mathbb{C}$ to be the category of étale spaces over B where B is assumed locally connected. For $\mathbb{X}$ we take the category of sets over $\pi_0 B$. The functor I is induced by π_0. The functor H is given by pullback by $B \to \pi_0 B$. The more general situation of coverings in a molecular topos, introduced by Barr and Diaconescu [1], is described in terms of an adjunction in [9], and so is also a special case of 4.2.

So in each of these cases we obtain a Van Kampen theorem for the pullback diagram (8): *the natural morphism*

$$\mathscr{F}(B) \to \mathscr{F}(B_1) \times_{\mathscr{F}(B_0)} \mathscr{F}(B_2)$$

is an equivalence of categories provided $B_1 \to B$, $B_2 \to B$ *are monomorphisms,* $B_0 = B_1 \times_B B_2$, *and* $B_1 + B_2 \to B$ *is an effective global-descent morphism.*

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